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"Time-Temperature Characteristics of Thin-Skinned Models as Affected by Thermocouple Variables"

by

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Part of the reason this report has been delayed is to allow the inclusion of the thesis of Paul Nelson, which is provided as Appendix I.

Several errors and omissions in the Third Semiannual Report have been noted and are mentioned throughout this report.

Continuation of Two-wire Problem

Inasmuch as it appears that an exact solution of the two-wire problem is beyond the scope of the present project, it is desirable to see if the solution already obtained for the case of two semi-infinite sheets butted to an infinite sheet (eq. 36, 37, and 38, Third Semiannual Report) may be used to approximate such a solution if only to indicate the importance of the various terms in the solution. Such an indication would be very helpful in the design of the shock tunnel models under consideration.

Solutions for the single simple wire and the single sheet junction have been compared in terms of the dimensional variables.

Single Simple Wire

$$T_1 = \frac{Q_1 \theta}{\rho_1 c_1 S}\left\{ 1 - \frac{\frac{r_0}{S}\sqrt{\frac{k_2 c_2 \rho_2}{k_1 c_1 \rho_1}}}{2 + \frac{r_0}{S}\sqrt{\frac{k_2 c_2 \rho_2}{k_1 c_1 \rho_1}}}\; 4 i^2 \operatorname{erfc} \frac{x}{2\sqrt{\alpha_1 \theta}} \right\}$$

where x is interpreted as the arc length along the skin from the junction.

Single Simple Sheet

$$T_1 = \frac{Q_1 \theta}{\rho_1 c_1 S_1}\left\{1 - \frac{S_2/S_1\sqrt{\frac{k_2 c_2 \rho_2}{k_1 c_1 \rho_1}}}{2 + S_2/S_1\sqrt{\frac{k_2 c_2 \rho_2}{k_1 c_1 \rho_1}}}\; 4\, i^2 \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha_1 \theta}}\right)\right\}$$

where x is the distance in the skin from the junction.

Comparison of these solutions indicate that they will be identical when the radius of the wire is equal to the thickness of the plane sheet. This suggests that the two sheet solutions may be used to understand the two wire problem by replacing the sheet thicknesses by the radii of the wires. We are not able to prove that this procedure leads to a correct solution, but based upon the physics of the problem we feel that such a solution is of value--at least along the line joining the two wires.

Page 7 of the previous report was incomplete and should have been

$$t_1 = \frac{1}{s^2} - \frac{e^{-\sqrt{s}(d/s_1 - x)}\,\beta_3(2+\beta_2)}{A s^2} + \frac{e^{-\sqrt{s}(2d/s_1 - x)}\,\beta_2\beta_3}{A s^2}$$

$$- \frac{e^{-\sqrt{s}(3d/s_1 - x)}\,K\beta_3(2+\beta_2)}{A s^2} + \frac{e^{-\sqrt{s}(4d/s_1 - x)}\,K\beta_2\beta_3}{A s^2}$$

$$- \frac{e^{-\sqrt{s}(5d/s_1 - x)}\,K^2\beta_3(2+\beta_2)}{A s^2} + \frac{e^{-\sqrt{s}(6d/s_1 - x)}\,K^2\beta_2\beta_3}{A s^2} + \cdots\cdots$$

$$- \frac{e^{-\sqrt{s}\,x}\,\beta_2}{s^2\,(2+\beta_2)} + \frac{e^{-\sqrt{s}(d/s_1 + x)}\,\beta_2\beta_3}{A s^2}$$

$$- \frac{e^{-\sqrt{s}(2d/s_1 + x)}\,\beta_2^2\beta_3}{A s^2(2+\beta_2)} + \frac{e^{-\sqrt{s}(3d/s_1 + x)}\,K\beta_2\beta_3}{A s^2} + \cdots\cdots .$$

Equation (37) on Page 9 contained an error in an exponent on K in the last line. (Correct by changing $3 \rightarrow 2$). Equations 36, 37 and 38 may be rewritten in the following form, allowing easier interpretation of the importance of each term:

for $x < 0$

$$\begin{aligned}
\tau_1(x,\phi) = \phi \Bigg\{ 1 &- \left(\frac{\beta_2}{2+\beta_2}\right) 4\, i^2\,\mathrm{erfc}\, \frac{-x}{2\sqrt{\phi}} \\
&- \left[\left(\frac{\beta_3}{2+\beta_3}\right) - \left(\frac{\beta_2}{2+\beta_2}\right)\left(\frac{\beta_3}{2+\beta_3}\right)\right] 4 \sum_{n=1}^{\infty} K^{n-1}\, i^2\,\mathrm{erfc}\, \frac{(2n-1)d/s_1 - x}{2\sqrt{\phi}} \\
&+ \left[\left(\frac{\beta_2}{2+\beta_2}\right)\left(\frac{\beta_3}{2+\beta_3}\right) - \left(\frac{\beta_2}{2+\beta_2}\right)^2\left(\frac{\beta_3}{2+\beta_3}\right)\right] 4 \sum_{n=1}^{\infty} K^{n-1}\, i^2\,\mathrm{erfc}\, \frac{2nd/s_1 - x}{2\sqrt{\phi}} \Bigg\}
\end{aligned} \tag{36}$$

for $0 \leq x \leq d/s_1$

$$\begin{aligned}
\tau_1(x,\phi) = \phi \Bigg\{ 1 &- \left(\frac{\beta_2}{2+\beta_2}\right) 4\, i^2\,\mathrm{erfc}\, \frac{x}{2\sqrt{\phi}} \\
&- \left(\frac{\beta_3}{2+\beta_3}\right) 4 \sum_{n=1}^{\infty} K^{n-1}\, i^2\,\mathrm{erfc}\, \frac{(2n-1)d/s_1 - x}{2\sqrt{\phi}} \\
&+ \left(\frac{\beta_2}{2+\beta_2}\right)\left(\frac{\beta_3}{2+\beta_3}\right) 4 \sum_{n=1}^{\infty} K^{n-1} \left[i^2\,\mathrm{erfc}\, \frac{(2n-1)d/s_1 + x}{2\sqrt{\phi}} + i^2\,\mathrm{erfc}\, \frac{2nd/s_1 - x}{2\sqrt{\phi}} \right] \\
&- \left(\frac{\beta_2}{2+\beta_2}\right)^2\left(\frac{\beta_3}{2+\beta_3}\right) 4 \sum_{n=1}^{\infty} K^{n-1}\, i^2\,\mathrm{erfc}\, \frac{2nd/s_1 + x}{2\sqrt{\phi}} \Bigg\}
\end{aligned} \tag{37}$$

For $s > d/s_1$

$$\tau_1(x,\phi) = \phi \left\{ 1 - \frac{\beta_3}{2+\beta_3} 4\, i^2 \operatorname{erfc} \frac{x - d/s_1}{2\sqrt{\phi}} \right.$$

$$- \left[\frac{\beta_2}{2+\beta_2} - \left(\frac{\beta_2}{2+\beta_2}\right)\left(\frac{\beta_3}{2+\beta_3}\right) \right] 4 \sum_{n=1}^{\infty} K^{n-1}\, i^2 \operatorname{erfc} \frac{x+(2n-2)d/s_1}{2\sqrt{\phi}}$$

$$+ \left[\left(\frac{\beta_2}{2+\beta_2}\right)\left(\frac{\beta_3}{2+\beta_3}\right) - \left(\frac{\beta_2}{2+\beta_2}\right)\left(\frac{\beta_3}{2+\beta_3}\right)^2 \right] 4 \sum_{n=1}^{\infty} K^{n-1}\, i^2 \operatorname{erfc} \frac{x+(2n-1)d/s_1}{2\sqrt{\phi}}$$

It is only recently that we have been aware of the similarity of these equations to those of transmission lines with multiple reflections. Further study of this similarity and its meaning may help in understanding the solutions obtained.

Several features of these equations are of interest.

1. The ratios $\frac{B_2}{2 + B_2}$ or $\frac{B_3}{2 + B_3}$ are always less than one and usually can be made of the order of 0.1 by selection of appropriate materials.

2. The value of $i^2\operatorname{erfc} y$ is 0.25 for $y = 0$, drops to less than 1% of this value for $y = 1.5$, and is tabulated as 0.000000 for $y = 3.0$.

3. Because of the occurrence of $\sqrt{\phi}$ in the denominator of the argument of $i^2\operatorname{erfc} y$ more terms will be needed to satisfactorily represent the temperature as the time increases.

As a consequence, if we are interested in the situation where $x = 0$ and we are willing to neglect terms which are approximately 1% of the first correction term ($\frac{B_2}{2 + B_2}$ $4\, i^2\operatorname{erfc}$ $\frac{-x}{2\sqrt{\phi}}$), then we may neglect any additional terms if $d/s_1 > 1.5\sqrt{\phi}$.

It should be noted that the first correction term, $\left(\frac{B_2}{2 + B_2} \; 4i^2\mathrm{erfc} \; \frac{-x}{2\sqrt{\phi}} \right)$, is exactly the term which would be expected for the one sheet in the absence of the second. The next correction term (considering n = 1) would be

$$\frac{B_3}{2 + B_3} \; 4i^2\mathrm{erfc} \; \frac{d/s_1 - x}{2\sqrt{\phi}} .$$

This is exactly the term to be expected as a contribution from the second sheet in the absence of the first if superposition were valid.

The following data present the values of T/ϕ at $x = 0$ (base of the constantan sheet) for the case of 0.0025" thick sheets of constantan and chromel fastened to a 0.004" thick copper skin for nondimensional times corresponding to approximately 0.1 milli-second and 4.1 milliseconds after start of heating. The value of zero spacing was obtained from the solution for two wires at a single point, while the value at infinity was calculated using the single sheet equation.

Values of Tau/PHI

d/s_1	$\phi = 1.0$	$\phi = 41.0$
0	0.87072	0.87072
4	0.92647	0.89933
8	0.92652	0.91497
12	0.92652	0.92228
16	0.92652	0.92515
20	0.92652	0.92614
24	0.92652	0.92643
28	0.92652	0.92650
32	0.92652	0.92652
∞	0.92652	0.92652

These data should indicate the nature of Tau/Phi for wires of a radius equal to the thickness of the sheets.

A solution at various values of x has been calculated for $d/s_1 = 16$ and is tabulated on Table I. This solution was made taking three terms in each of the summations. In no case did the third term contribute to the solution.

	$\phi = 1.0$	$\phi = 11.0$	$\phi = 21.0$	$\phi = 31.0$	$\phi = 41.0$
x = -8.0	Tau = 1.00000	Tau = 10.9784	Tau = 20.8637	Tau = 30.6673	Tau = 40.4102
x = -6.4	1.00000	10.9479	20.7608	30.4843	40.1497
x = -4.8	.99999	10.8841	20.5954	30.2220	39.7930
x = -3.2	.99962	10.7617	20.3447	29.8564	39.3253
x = -1.6	.99177	10.5459	19.9754	29.3610	38.7131
x = 0.0	.92652	10.1917	19.4535	28.7023	37.9313
x = 1.6	.99177	10.5455	19.9689	29.3345	38.6529
x = 3.2	.99962	10.7610	20.3284	29.7955	39.1960
x = 4.8	.99999	10.8816	20.5615	30.1133	39.5730
x = 6.4	1.00000	10.9406	20.6950	30.3026	39.8079
x = 8.0	1.00000	10.9595	20.7439	30.3761	39.8990

Single Wire with Added Capacity and Resistance

A minus sign was omitted from the bottom equation on page 13. The last line of the equation at the bottom of page 16 should have read

$$\left(\frac{c_2'\psi^2 + 2c_2'\phi}{2}\right)\operatorname{erfc}\frac{\psi}{2\sqrt{\phi}} - \frac{c_2'\phi^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}\psi e^{-\frac{\psi^2}{4\phi}} .$$

Equation 40 should then have been

$$\begin{aligned}\tau_1(\psi,\phi) = \phi &- \left[(a_1 - b_1\psi)\frac{L}{C} + (a_2 - b_2\psi)\frac{1}{C} + \frac{c_2'}{2C}\psi^2 + \frac{c_2'\phi}{C}\right]\operatorname{erfc}\frac{\psi}{2\sqrt{\phi}} \\ &- \left[\left(\frac{d_2}{C} + \frac{c_1'L}{C}\right)e^{G(G\phi+\psi)}\right]\operatorname{erfc}\left(\frac{\psi}{2\sqrt{\phi}} + G\sqrt{\phi}\right) \\ &- \left[\left(\frac{e_2}{C} + \frac{d_1L}{C}\right)e^{H(H\phi+\psi)}\right]\operatorname{erfc}\left(\frac{\psi}{2\sqrt{\phi}} + H\sqrt{\phi}\right) \\ &- \left[\frac{2\phi^{\frac{1}{2}}b_2}{\pi^{\frac{1}{2}}C} - \frac{\phi^{\frac{1}{2}}\psi c_2'}{\pi^{\frac{1}{2}}C} + \frac{2\phi^{\frac{1}{2}}b_1L}{\pi^{\frac{1}{2}}C}\right]e^{-\frac{\psi^2}{4\phi}} .\end{aligned}$$

The coefficients of equation (40) have been evaluated in terms of the parameters: L, N and B. The resulting equation has been put into a form containing the term

$$\left(\frac{\beta}{2+\beta}\right)\phi 4 L^2 \operatorname{erfc}\frac{\psi}{2\sqrt{\phi}} .$$

This is the term expressing the magnitude of the correction in the event of no added capacity or resistance. Thus, other terms appearing in the expression represent second order corrections occurring due to such capacity or resistance.

$$\tau_1(\psi,\phi) = \phi - \left(\frac{\beta}{2+\beta}\right)\phi\, 4L^2 \operatorname{erfc}\frac{\psi}{2\sqrt{\phi}}$$

$$+\left\{\left(\frac{\beta}{2+\beta}\right)^2(2)\left(2NL+\frac{L^2}{2}\right)-\left(\frac{\beta}{2+\beta}\right)^3(L+2N)^2\right\}\operatorname{erfc}\frac{\psi}{2\sqrt{\phi}}$$

$$-\left\{\left(\frac{\beta}{2+\beta}\right)2L-2(L+2N)\left(\frac{\beta}{2+\beta}\right)^2\right\}\frac{\phi^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}\,e^{\frac{-\psi^2}{4\phi}}$$

$$-\psi\left[\left(\frac{\beta}{2+\beta}\right)^2(L+2N)-\left(\frac{\beta}{2+\beta}\right)L\right]\operatorname{erfc}\frac{\psi}{2\sqrt{\phi}}$$

$$-\frac{1}{2NL\,G^2(G-H)}\left[\frac{1}{G}-L\right]e^{G(G\phi-\psi)}\operatorname{erfc}\left(\frac{\psi}{2\sqrt{\phi}}+G\sqrt{\phi}\right)$$

$$-\frac{1}{2NL\,H^2(H-G)}\left[\frac{1}{H}-L\right]e^{H(H\phi-\psi)}\operatorname{erfc}\left(\frac{\psi}{2\sqrt{\phi}}+H\sqrt{\phi}\right)$$

Detached

As is pointed out by Paul Nelson in Appendix I, the above solution is valid if G and H are real or complex but unequal. Mr. Nelson has also formulated solutions which can be more readily used for the unequal complex case and for the case where G = H. Future work will attempt to define more clearly the probable use of these equations in practice.